NON SELF-SIMILAR SETS IN $[0,1]^N$ OF ARBITRARY DIMENSION

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Abstract. We consider $[0,1]^N$, the unit cube of $\mathbb{R}^N$, $N \geqslant 1$. Let $\mathcal{S} = \{S_1, \dots, S_M\}$ be a finite set of contraction maps from X to itself. A non-empty subset E of X is an *attractor* (or an *invariant set*) for the iterated function system (IFS) $\mathcal{S}$ if $E = \cup_{i=1}^M S_i(E)$.
We construct, for each $s \in]0, N]$, a nowhere dense perfect set E contained in $[0,1]^N$, with Hausdorff dimension s, which is not an attractor for any iterated function system composed of weak contractions from $[0,1]^N$ to itself.

1. Introduction

Let $X = (X, d)$ be a complete metric space. Let $\mathcal{S} = \{S_1, \dots, S_M\}$ be a finite set of contraction maps from X to itself. A subset E of X is an *attractor* (or an *invariant set*) for the iterated function system (IFS) $\mathcal{S}$ if $E = \cup_{i=1}^M S_i(E)$ ([11], [14]). Following [14] we write $\cup_{i=1}^M S_i(E) = \mathcal{S}(E)$. By $(\mathcal{K}(X), \mathcal{H}) = \mathcal{K}(X)$ we denote the metric space comprised of the non-empty compact subsets of X endowed with the Hausdorff metric. It turns out that, for any given IFS $\mathcal{S}$, there exists a unique non-empty compact set $E \subseteq X$ such that $E = \mathcal{S}(E)$.
This study is motivated by recent research concerning Cantor sets, iterated functions systems and the structure of attractors ([1], [4], [5], [6], [10], [11], [12], [13], [14], [17], [18], [19]).
Let

$$\mathcal{T} = \{E \in \mathcal{K}(X) : E = \mathcal{S}(E); \mathcal{S} \textit{ a finite collection of contraction maps}\}$$

to be the set of attractors for contractive systems defined on X. It turns out that, in the case when X is compact, $\mathcal{T}$ is always an F_σ subset of $\mathcal{K}(X)$ ([8] and [9]). The space $\mathcal{K}(X)$ is complete [3], so it is appropriate to use Baire Category Theorem and to investigate typical (or generic) properties (with respect to Baire Category). The term typical (or generic) indicates

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that the collection of sets having the property under consideration has first category complement in the complete metric space $\mathcal{K}(X)$, hence it is "large" with respect to Baire classification.
In the case that $X = [0,1]^N$, $N \geqslant 1$, the set $\mathcal{T}$ is a (Baire) first category set, that is the typical (or generic) compact subset of $[0,1]^N$ is not an attractor for any system of contractions [9]. More precisely, the typical $E \in \mathcal{K}([0,1]^N)$ has the following properties

(1) E is perfect, nowhere dense and totally disconnected, that is E is a Cantor space,
(2) $E \subset \mathbb{I}\mathbb{R}$, where $\mathbb{I}\mathbb{R} = \{(a_1, \ldots, a_N) \in X : a_i \in [0,1] \setminus \mathbb{Q},\ \textit{for all}\ 1 \leqslant i \leqslant N\}$, that is E consists of points whose coordinates are irrational,
(3) for each $s > 0$, $\mathcal{H}^s(E) = 0$, and
(4) E is not invariant with respect to any iterated function system comprised of contractions.

In [7] it is given, for each $s \in]0,1]$, an example of a nowhere dense perfect set E contained in the unit interval with $dim_{\mathcal{H}}(E) = s$, which is not an attractor for any iterated function system composed of weak contractions. This result answers, in the case when $N = 1$, a problem posed by Zoltán Buczolich at the Summer Symposium in Real Analysis XXXIX (June 8-13), 2015, St. Olaf College, Northfield, MN). In this note, we give a general construction, in any dimension N, $N \geqslant 1$. For any $N \geqslant 1$, for each $s \in]0, N]$, we give a construction of a nowhere dense perfect set contained in $[0,1]^N$, with Hausdorff dimension s, and such that $E \neq \mathcal{S}(E)$ whenever $\mathcal{S} = \{S_1, \ldots, S_M\}$, and $d(S_i(x), S_i(y)) < d(x,y)$, for $i = 1, \ldots, M$.

2. Notations and Preliminary results

Let $(X, d) = X$ be a metric space.
Let A and B be subsets of X. We let $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$, and by $|A|$ we denote the diameter of A.

Let $(X, d) = X$ be a complete metric space. As in [14], let $\mathcal{B}(X)$ be the class of non-empty closed and bounded subsets of X. We endow $\mathcal{B}(X)$ with the Hausdorff metric $\mathcal{H}$ given by $\mathcal{H}(E,F) = inf\{\delta > 0 : E \subset B_\delta(F), F \subset B_\delta(E)\}$. This space is complete. In the case that X is also compact, then $\mathcal{B}(X) = \mathcal{K}(X)$, where $\mathcal{K}(X) = (\mathcal{K}(X), \mathcal{H})$ is the class of non-empty compact subsets of X, and $\mathcal{K}(X)$ is also compact [3].

Contractions and weak contractions
Let $(X, d) = X$ be a metric space. Let $f : X \to X$. We define the *Lipschitz constant* of f

$$Lipf := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)} < 1.$$

If $Lipf < 1$, then we say that $f : X \to X$ is a *contraction*.

We say that $f : X \to X$ is a *weak contraction* if, for each x, y in X, $x \neq y$,

$$d(f(x), f(y)) < d(x, y).$$

Of course, each contraction is a weak contraction but the converse is not, in general, true.

Hausdorff measure and Hausdorff dimension

Let $N \in \mathbb{N}$. Suppose F is a subset of $\mathbb{R}^N$ and s is a non-negative number. For any $\delta > 0$, set

$$\mathcal{H}^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \textit{ is a } \delta\textit{-cover of } F\}.$$

Then $\mathcal{H}^s = \lim_{\delta\to\infty} \mathcal{H}^s_\delta$ defines a measure on the Borel sets in $\mathbb{R}^N$, generally referred to as the s-dimensional Hausdorff measure ([11], [15]). (We say that $\{U_i\}$ is a δ-*cover* of F if: $(i) F \subseteq \cup_{i=1}^{\infty} U_i$ and (ii) $0 \leqslant |U_i| \leqslant \delta$.)

The Hausdorff dimension, $dim_{\mathcal{H}}(F)$, is defined as

$$dim_{\mathcal{H}}(F) = \inf\{s \geqslant 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$

(taking the supremum of the empty set to be 0).

General Cantor sets

We recall that a topological space is a *Cantor space* if it is homeomorphic to the Cantor ternary set. It follows that a topological space is a *Cantor space* if and only if it is non-empty, perfect, compact, totally disconnected, and metrizable.

General Cantor sets are all Cantor spaces. These sets are generalizations of the classical Cantor set. A precise definition follows.

Definition 1. ([2]; [20]) A set E is said to be a general Cantor set if and only if it can be expressed in the form

$$E = \cap_{n=1}^{\infty} \cup_{i_1,\dots,i_n=1}^{K} \Theta_{i_1\dots i_n},$$

where $K \geqslant 2$ is an integer and where the $\Theta_{i_1\dots i_n}$ are connected, compact sets satisfying

(1) $\Theta_{i_1,\dots,i_n} \supset \Theta_{i_1\dots,i_n i_{n+1}}$
(2) $\Theta_1, \dots, \Theta_K$ are mutually disjoint,
(3) there exists a constant A, $0 < A < 1$, such that

$$|\Theta_{i_1\dots,i_n i_{n+1}}| \geqslant A|\Theta_{i_1\dots i_n}| \qquad (i_{n+1} = 1, \dots, K),$$

(4) there exists a constant B, $0 < B < 1$, such that for $s \neq t$,

$$d(\Theta_{i_1\dots i_n s}, \Theta_{i_1\dots i_n t}) \geqslant B|\Theta_{i_1\dots i_n}|$$

.

[2] A general Cantor set is called a *spherical Cantor set* if and only if, for each choice of $i_1, \ldots, i_n$, $\Theta_{i_1 \ldots, i_n}$ is an N-dimensional sphere.
Since we can approximate spheres by cubes and viceversa, we can replace in the above definition "N-dimensional sphere" with "N-dimensional cube".
Clearly, in the case $n = 1$, the two definitions, of a general Cantor and of a spherical Cantor set, coincide.

Theorem 2. *(*[16], [2]*) Fix $N \in \mathbb{N}$. For each $s \in]0, N[$ there exists a general Cantor set in $[0,1]^N$ with $dim_{\mathcal{H}}(E) = s$ and $0 < \mathcal{H}^s(E) < \infty$.*

Theorem 3. *(*[16], [2]*) Let $N \geqslant 2$. For each $s \in]0, N[$ there exists a spherical Cantor set in $[0,1]^N$ with $dim_{\mathcal{H}}(E) = s$ and $0 < \mathcal{H}^s(E) < \infty$.*

Dyadic Cubes
We recall that the *dyadic cubes* are a collection of cubes in $\mathbb{R}^N$ of different sizes or scales such that the set of cubes of each scale partitions $\mathbb{R}^N$ and each cube in one scale may be written as a union of cubes of a smaller scale.
Dyadic cubes may be constructed as follows: for each $k = 0, \pm 1, \pm 2, \ldots$, let Δ_k be the set of cubes in $\mathbb{R}^N$ of side-length $\frac{1}{2^k}$ and corners in the set

$$2^{-k}\mathbb{Z}^N = \{2^{-k}(v_1, \ldots, v_N) : v_j \in \mathbb{Z}\}$$

and let Δ be the union of all Δ_k.
The most important features of these cubes are the folowing:

a. For each integer k, Δ_k partitions $\mathbb{R}^N$.
b. All cubes in Δ_k have the same side-length, namely $\frac{1}{2^k}$.
c. If the interior of two cubes Q and R in Δ_k have nonempty intersection, then either Q is contained in R or R is contained in Q.
d. Each Q in Δ_k maybe written as a union of 2^N cubes in Δ_{k+1} with disjoint interiors.

Then, clearly, for each $k = 0, 1, 2, \ldots$, we have

$$[0,1]^N = \cup_{Q \in \Delta_k} Q \cap [0,1]^N,$$

where $Q \cap [0,1]^N \neq \emptyset$ if and only if Q has corners in the set

$$2^{-k}\{0, 1, \ldots, 2^k\}^N = \{2^{-k}(v_1, \ldots, v_N) : v_j \in \{0, 1, \ldots, 2^k\}\}$$

Remark 4. Clearly, in the case when $N = 1$, we deal with dyadic intervals. Our construction works for the general case $N \geqslant 1$ so, even if we always use the word cube, for $N = 1$ it is appropriate to talk about dyadic intervals.

3. Construction of non self-similar Cantor sets

Theorem 5. *Let $N \in \mathbb{N}$. For each $s \in]0, N]$ there exists a subset E of $[0,1]^N$, nowhere dense and perfect, with $dim_{\mathcal{H}}(E) = s$, that is not the attractor for any iterated function system composed of weak contractions from $[0,1]^N$ to itself.*

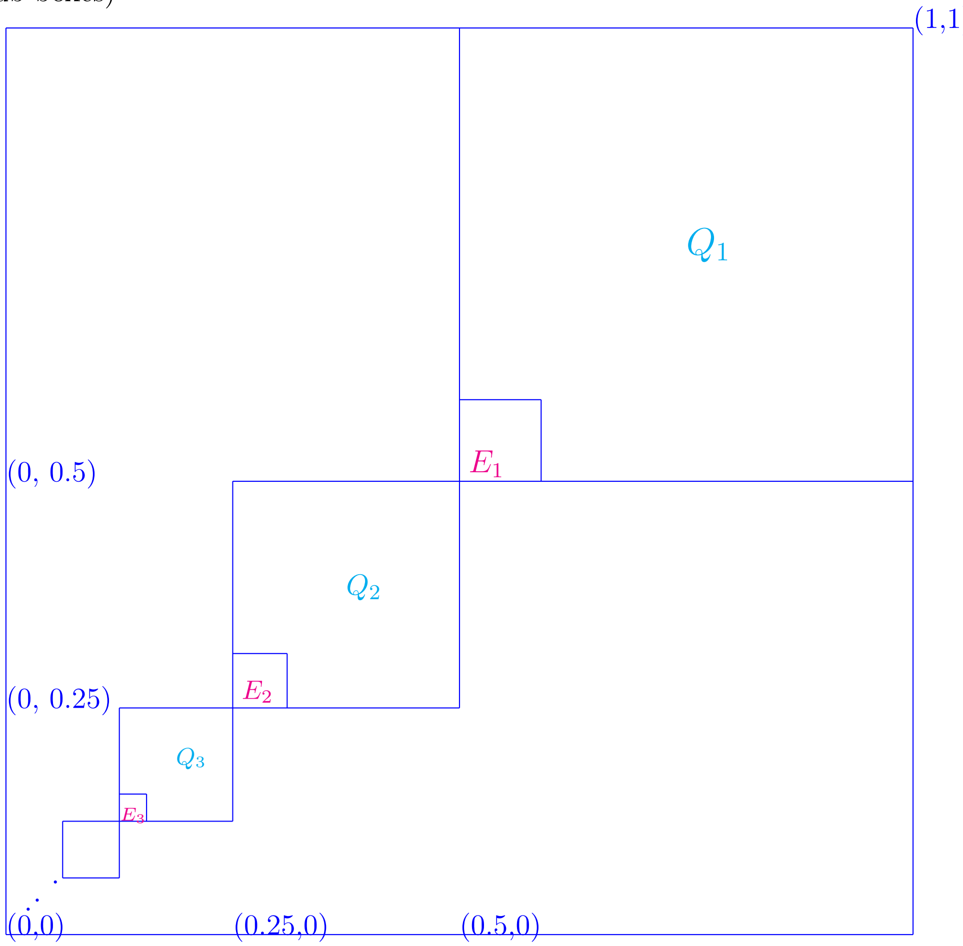
Proof. ($\bullet$) We start by defining the set E. We define E as

$$E = \{0\} \cup \{\cup_{k=1}^{\infty} E_k\},$$

where the sets E_k are taken so that, for each k:

(a) E_k is contained in $Q_k \in \Delta_k$, where $Q_k = [\frac{1}{2^k}, \frac{1}{2^{k-1}}]^N$,
(b) $|E_k| = c_N \frac{1}{2^k}$, where c_N is any fixed constant with $0 < c_N < \frac{1}{2}\frac{\sqrt{N}}{1+\sqrt{N}}$,
(c) the point $\underline{x} = (x_1, \ldots, x_n)$ with $x_i = \frac{1}{2^k}$, for each $k \in \mathbb{N}$, is in E_k,
(d) $s_k = dim_H(E_k) = s - \frac{s}{k+1}$; hence, $\{s_k\}$ is an increasing sequence with $\lim_k s_k = \sup_k s_k = s$, and
(e) $0 < \mathcal{H}^{s_k}(E_k) < N$.

Figure: construction for $N = 2$ (the E_k's are contained in the small sub-boxes)



Clearly, from the construction above, it follows immediately that

(i) $d(E_k, E \setminus E_k) > |E_k|$,
(ii) $dim_{\mathcal{H}}(E) = s$ and $\mathcal{H}^s(E) = 0$, and

(iii) if $m > t$, then $\mathcal{H}^{s_m}(f(E_t) \cap E_m) = 0$, for any Lipschitz map f.

(••) We now show that E cannot be the attractor (invariant set) of any system of weak contractions from $[0,1]^N$ to itself.
Let $f : E \to E$ be a weak contraction. We distinguish two cases: 1) $f(0) = 0$ and 2) $f(0) \neq 0$.
Case 1. Suppose $f(0) = 0$. Then, it follows that, for any k, $f(E_k) \subset E \setminus \cup_{i=1}^{k} E_i$. In fact, there exists $x \in E_k$ such that $d(0,x) = d(0,E_k)$. Hence, as f is a week contraction, $d(0,f(x)) < d(0,x) = d(0,E_k)$. Therefore, $f(x) \in E \setminus \cup_{i=1}^{k} E_i$. The conclusion follows from the observation that

$$|f(E_k)| < |E_k| < d(E_k, E \setminus E_k).$$

Case 2. Suppose $f(0) \neq 0$. Then there exists $m_0 \in \mathbb{N}$ such that $\mathcal{H}^{s_m}(f(E) \cap E_m) = 0$ whenever $m > m_0$. In fact, there exist k and $x \in E_k$ with $x = f(0)$. By the continuity of f, there exists $n_0 \in \mathbb{N}$ such that $f(E_m) \subseteq E_k$ whenever $m > n_0$.
Consider $L = \cup_{i=1}^{n_0} E_i$. Then $\mathcal{H}^{s_m}(f(L) \cap E_m) = 0$ for any $m > n_0$, and $\mathcal{H}^{s_m}(f(E) \cap E_m) = 0$ for any $m > m_0 = \max\{n_0, k\}$.

Let $\mathcal{S} = \{S_1, \dots, S_M\}$ be a finite collection of weak contractions from $[0,1]^N$ to itself. We write $\mathcal{S}$ as a disjoint union, $\mathcal{S} = \mathcal{S}_\star \cup \mathcal{S}_{\star\star}$, where $\mathcal{S}_\star$ consists of the S_i's in $\mathcal{S}$ with $S_i(0) = 0$ and, hence, $\mathcal{S}_{\star\star}$ consists of the S_i's in $\mathcal{S}$ with $S_i(0) \neq 0$.
If $S_i \in \mathcal{S}_\star$, then, for each k, ${S_i}^{-1}(E_k) \subseteq \cup_{j=1}^{k-1} E_j$. Thus, $\mathcal{H}^{s_k}(S_i(E) \cap E_k) = 0$.
If $S_i \in \mathcal{S}_{\star\star}$, there exists m_i such that $\mathcal{H}^{s_m}(S_i(E) \cap E_m) = 0$ for any $m > m_i$.
Let $\overline{n} = \max\{m_i : S_i \in \mathcal{S}_{\star\star}\}$. If we fix any E_k, $k > \overline{n}$, then, for each $S_i \in \mathcal{S}_{\star\star}$, $\mathcal{H}^{s_k}(S_i(E) \cap E_k) = 0$. Hence, if it was $E = \cup_{i=1}^{M} S_i(E)$, we would have

$$\begin{aligned} 0 &< \mathcal{H}^{s_k}(E_k) = \mathcal{H}^{s_k}(E \cap E_k) \\ &= \mathcal{H}^{s_k}((\cup_{i=1}^{M} S_i(E)) \cap E_k) \\ &\leqslant \sum_{i=1}^{t} \mathcal{H}^{s_k}(S_i(E) \cap E_k) = 0. \end{aligned}$$

Hence it must be $\mathcal{S}(E) \neq E$. □

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